J. Differential Equations 251 (2011) 1505–1525

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Critical periods of perturbations of reversible rigidly isochronous centers ☆

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ARTICLE INFO

Article history:
Received 22 April 2010
Revised 7 May 2011
Available online 28 May 2011

Keywords:
Polynomial system
Critical period
Bifurcation
Isochronous center

ABSTRACT

In this paper we discuss bifurcation of critical periods in an m-th degree time-reversible system, which is a perturbation of an n-th degree homogeneous vector field with a rigidly isochronous center at the origin. We present period-bifurcation functions as integrals of analytic functions which depend on perturbation coefficients and reduce the problem of critical periods to finding zeros of a judging function. This procedure gives not only the number of critical periods bifurcating from the period annulus but also the location of these critical periods. Applying our procedure to the case $n = m = 2$ we determine the maximum number of critical periods and their location; to the case $n = m = 3$ we investigate the bifurcation of critical periods up to the first order in ε and obtain the expression of the second period-bifurcation function when the first one vanishes.

1. Introduction

We consider the two-dimensional analytic real differential system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1.1}$$

☆ Supported by NSFC #10825104, NSFC Tianyuan Fund #10926045, SRFDP #20090181120082 and #200806100002 of China, the China and Slovenia Science Cooperation Project ([2007] 160 and BI-CN/09-11-011), the Slovene Human Resources and Scholarship Fund and the Slovenian Research Agency (ARRS).

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.05.022

which has a nondegenerate (or elementary) center-focus at the origin $O=(0,0)$, that is, the Jacobian matrix of $(P(x,y),Q(x,y))$ at O has a pair of conjugate imaginary eigenvalues. There are few main directions in the current research regarding to the studying the behavior of solutions of system (1.1) near O. One is the so-called *center problem* or the *stability problem*, that is, the problem to distinguish systems (1.1) with a center at O from those with a focus. If the center problem is solved then the next arising problem is to investigate bifurcations of limit cycles when centers are destroyed. It is closely related to the Hilbert's 16th problem [19], which is the problem to determine the maximum number of limit cycles of system (1.1) when both P and Q are polynomials of degree n. Another aspect of studying centers is the so-called *isochronicity problem*, that is, identifying isochronous center in the family of weak centers of finite order in (1.1). In this direction, another interesting problem is to consider the critical points of the *period function* of system (1.1), called *critical periods*, which is related to subharmonic bifurcations [18].

Assume that O is a center of system (1.1) and let $T(\rho)$ be the period of the periodic orbit passing through the point $(\rho,0)$, called the *period function* of (1.1). $T(\rho_*)$ is called a *critical period* of the system if $T(\rho_*)$ is a critical value of the period function, i.e., $T'(\rho^*)=0$. The research on period functions has been developed extensively on, e.g., monotonicity [1,5,8–10,14,26,27], finiteness of critical periods [6,21] and isochronicity [3,20,22]. Local bifurcation of critical periods [7], which concerns how many critical periods can arise near the center, is also an important problem in the study of period functions and many results are obtained for polynomial differential systems such as cubic homogeneous systems [23,24], reversible cubic systems [29], reduced Kukles system [25], Liénard systems [30] and generalized Lotka–Volterra systems [28]. Another aspect in research is the persistence of isochronicity (i.e., the period function $T(\rho)$ persists to be a constant) under perturbations. All nonlinear isochronous systems can be regarded as perturbations of the linear isochronous system which persist isochronicity such as homogeneous polynomial perturbations [2]. In [4,29] the persistence of isochronicity of quadratic isochronous systems is discussed under cubic time-reversible perturbations and all conditions for the persistence are obtained.

In recent years more and more attentions are paid to the global one, especially the bifurcation of critical periods from isochronous vector fields. Consider a perturbation of system (1.1)

$$\dot{x}=P(x,y)+\varepsilon\tilde{P}(x,y),\qquad \dot{y}=Q(x,y)+\varepsilon\tilde{Q}(x,y),\tag{1.2}$$

where ε is a sufficiently small positive number and O is a center and additionally an isochronous center when $\varepsilon=0$. As in [11–13,15,16], the period function $T(\rho,\varepsilon)$ of system (1.2) can be written in the expansion

$$T(\rho,\varepsilon)=T_0+\sum_{i=1}^{+\infty}T_i(\rho)\varepsilon^i,\tag{1.3}$$

where T_0 is a constant, which additionally equals 2π when the Jacobian matrix of $(P(x,y),Q(x,y))$ at O has a pair of conjugate imaginary eigenvalues $\pm\mathrm{i}$. For convenience, we call $T_i(\rho)$ the i-th *period-bifurcation function* (PBF for short) and say that $T(\rho,\varepsilon)$ is *vanishing of k-th order* if $T_1(\rho)\equiv\cdots\equiv T_{k-1}(\rho)\equiv 0$ and $T_k(\rho)\not\equiv 0$. Cima et al. [11, Theorem 2] proved that if $T(\rho,\varepsilon)$ is vanishing of k-th order and ρ^* is a simple zero of the k-th PBF T_k then there is a unique $\rho^*(\varepsilon)$ which tends to ρ^* as ε tends to 0 and satisfies $T'(\rho^*(\varepsilon),\varepsilon)=0$. The number of critical periods is discussed in [11] for perturbations of the linear isochronous vector field, i.e.,

$$\dot{x}=-y+\sum_{i=1}^{m}\varepsilon^i\tilde{P}^{(i)}(x,y),\qquad \dot{y}=x+\sum_{i=1}^{m}\varepsilon^i\tilde{Q}^{(i)}(x,y),\tag{1.4}$$

where $\tilde{P}^{(i)}$'s and $\tilde{Q}^{(i)}$'s are polynomials of degree n starting from degree 2. These results were also applied in [11] to potential, reversible and Liénard centers separately to give the maximum number of critical periods. The same problems were also discussed in [15,16] for perturbations of some nonlinear isochronous vector fields, e.g., quadratic isochronous vector fields.

In this paper we consider bifurcation of critical periods for the m-th degree time-reversible perturbation of n-th degree homogeneous time-reversible vector fields with a rigidly isochronous center at O (i.e., $\dot{\theta} \equiv 1$ in the polar coordinates, as defined in [3]). This system can be expressed as

$$\dot{x} = -y + P_n(x, y) + \varepsilon \sum_{i=2}^{m} \tilde{P}_i(x, y), \qquad \dot{y} = x + Q_n(x, y) + \varepsilon \sum_{i=2}^{m} \tilde{Q}_i(x, y), \tag{1.5}$$

where $m, n \geqslant 2$, $0 < \varepsilon \ll 1$,

$$P_n(x, y) = \sum_{j=1}^{[\frac{n}{2}]} a_{n+1-2j,2j-1} x^{n+1-2j} y^{2j-1}, \qquad \tilde{P}_i(x, y) = \sum_{j=1}^{[\frac{i+1}{2}]} b_{i+1-2j,2j-1} x^{i+1-2j} y^{2j-1},$$

$$Q_n(x, y) = \sum_{j=1}^{[\frac{n}{2}]} a_{n+1-2j,2j-1} x^{n-2j} y^{2j}, \qquad \tilde{Q}_i(x, y) = \sum_{j=1}^{[\frac{i+2}{2}]} c_{i+2-2j,2j-2} x^{i+2-2j} y^{2j-2},$$

$[\alpha]$ denotes the greatest integer not greater than the real α, the coefficients $a_{i,j}$, $b_{i,j}$, $c_{i,j}$ are all real and some $a_{n+1-2i,2i-1}$'s do not vanish. Note that after an appropriate change of coordinates every n-th degree homogeneous time-reversible system can be written in the form

$$\dot{x} = -y + \sum_{j=1}^{[\frac{n+1}{2}]} A_{n+1-2j,2j-1} x^{n+1-2j} y^{2j-1}, \qquad \dot{y} = x + \sum_{j=1}^{[\frac{n+2}{2}]} B_{n+2-2j,2j-2} x^{n+2-2j} y^{2j-2}. \tag{1.6}$$

Solving the equation $\dot{\theta} \equiv 1$ for system (1.6), we see that every n-th degree homogeneous time-reversible system with a rigidly isochronous center can be reduced to the form $\dot{x} = -y + P_n(x, y)$, $\dot{y} = x + Q_n(x, y)$. Clearly, the time-reversibility guarantees that O is a center of system (1.5) no matter whether $\varepsilon = 0$ or not. For convenience, we call $b_{i,j}$'s and $c_{i,j}$'s in system (1.5) the *perturbation coefficients*. We will give expressions of PBFs in the form of integrals of analytic functions which depend on those perturbation coefficients so as to reduce the problem of critical periods to finding zeros of a judging function. The judging function enables us not only to determine the number of critical periods but also to find the location of those critical periods. In Section 3, applying our method to the case $n = m = 2$, we not only obtain the same number of critical periods as in [15,16] but also determine their location accurately. We also apply our method to the case $n = m = 3$ in Section 4 and prove that at most two critical periods occur up to first order in ε and their location can be determined immediately as the perturbation coefficients are given.

2. Computations of PBFs and bifurcations

With the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ system (1.5) can be written in the form

$$\begin{cases} \dot{r} = r^n G_0(\theta) + \varepsilon \displaystyle\sum_{i=1}^{m-1} r^{i+1} G_i(\theta), \\ \dot{\theta} = 1 + \varepsilon \displaystyle\sum_{i=1}^{m-1} r^i S_i(\theta), \end{cases} \tag{2.1}$$

where

$$G_0(\theta)=\sum_{j=0}^{[\frac{n}{2}]}(a_{n-1-2j,2j+1}+a_{n+1-2j,2j-1})\cos^{n-2j}\theta\sin^{2j+1}\theta, \tag{2.2}$$

$$G_i(\theta)=\begin{cases}\sum_{j=1}^{[\frac{i+2}{2}]}(b_{i+2-2j,2j-1}+c_{i+3-2j,2j-2})\cos^{i+3-2j}\theta\sin^{2j-1}\theta, & \text{as } i \text{ is even},\\ \sum_{j=1}^{[\frac{i+2}{2}]}(b_{i+2-2j,2j-1}+c_{i+3-2j,2j-2})\cos^{i+3-2j}\theta\sin^{2j-1}\theta & \\ \quad +c_{0,i+1}\sin^{i+2}\theta, & \text{as } i \text{ is odd},\end{cases} \tag{2.3}$$

$$S_i(\theta)=\begin{cases}\sum_{j=0}^{[\frac{i+1}{2}]}(c_{i+1-2j,2j}-b_{i+2-2j,2j-1})\cos^{i+2-2j}\theta\sin^{2j}\theta & \\ \quad -b_{0,i+1}\sin^{i+2}\theta, & \text{as } i \text{ is even},\\ \sum_{j=0}^{[\frac{i+1}{2}]}(c_{i+1-2j,2j}-b_{i+2-2j,2j-1})\cos^{i+2-2j}\theta\sin^{2j}\theta, & \text{as } i \text{ is odd},\end{cases} \tag{2.4}$$

and $a_{n+1,-1}=a_{n-1-2[n/2],2[n/2]+1}=0$. Clearly, $\dot\theta\equiv 1$ when $\varepsilon=0$, implying that the unperturbed system $(1.5)|_{\varepsilon=0}$ has an isochronous center at O, called a *rigidly* or *uniformly* isochronous center as in [3]. Therefore, system (1.5) is actually a polynomial perturbation of a nonlinear isochronous center.

Theorem 2.1. *For system* (1.5),

$$T_1(\rho)=-\sum_{i=1}^{m-1}\int_0^{2\pi}\frac{\rho^i S_i(\theta)}{(1+(1-n)\rho^{n-1}\int_0^\theta G_0(\alpha)\,d\alpha)^{\frac{i}{n-1}}}\,d\theta, \tag{2.5}$$

$$T_2(\rho)=\int_0^{2\pi}\left(\left(\sum_{i=1}^{m-1}r_0^i(\theta,\rho)S_i(\theta)\right)^2-\sum_{i=1}^{m-1}ir_0^{i-1}(\theta,\rho)r_1(\theta,\rho)S_i(\theta)\right)d\theta, \tag{2.6}$$

where $r_0(\theta,\rho)$ and $r_1(\theta,\rho)$ are given by

$$r_0(\theta,\rho)=\left(\rho^{1-n}+(1-n)\int_0^\theta G_0(\alpha)\,d\alpha\right)^{1/(1-n)}, \tag{2.7}$$

$$r_1(\theta,\rho)=\int_0^\theta\left\{\sum_{i=1}^{m-1}r_0^{i+1}(\alpha,\rho)G_i(\alpha)-G_0(\alpha)\sum_{i=1}^{m-1}r_0^{n+i}(\alpha,\rho)S_i(\alpha)\right\}$$
$$\cdot\exp\left(\int_\alpha^\theta nr_0^{n-1}(\beta,\rho)G_0(\beta)\,d\beta\right)d\alpha. \tag{2.8}$$

Moreover, if n is odd, then

$$T_1(\rho)=-\sum_{k=0}^{[\frac{m-1}{2}]}\int_0^{2\pi}\frac{\rho^{2k}S_{2k}(\theta)}{(1+(1-n)\rho^{n-1}\int_0^\theta G_0(\alpha)\,d\alpha)^{\frac{2k}{n-1}}}\,d\theta, \tag{2.9}$$

where $S_0(\theta)\equiv 0$.

Proof. From (2.1) we get the differential equation

$$\frac{dr}{d\theta}=\frac{r^n G_0(\theta)+\varepsilon\sum_{i=1}^{m-1} r^{i+1}G_i(\theta)}{1+\varepsilon\sum_{i=1}^{m-1} r^i S_i(\theta)}. \tag{2.10}$$

Let $r(\theta,\varepsilon;\rho)$ be the solution of Eq. (2.10) satisfying the initial condition $r(0,\varepsilon;\rho)=\rho$. It can be written as the series

$$r(\theta,\varepsilon;\rho)=\sum_{i=0}^{+\infty} r_i(\theta,\rho)\varepsilon^i, \tag{2.11}$$

where $r_0(0,\rho)=\rho$ and $r_i(0,\rho)\equiv 0$ for $i\geqslant 1$. Substituting (2.11) in (2.10) and comparing the coefficients, we obtain that

$$\frac{d}{d\theta}r_0(\theta,\rho)=r_0^n(\theta,\rho)G_0(\theta), \tag{2.12}$$

$$\begin{aligned}\frac{d}{d\theta}r_1(\theta,\rho)={}&nr_0^{n-1}(\theta,\rho)G_0(\theta)r_1(\theta,\rho)+\sum_{i=1}^{m-1} r_0^{i+1}(\theta,\rho)G_i(\theta)\\&-G_0(\theta)\sum_{i=1}^{m-1} r_0^{n+i}(\theta,\rho)S_i(\theta).\end{aligned} \tag{2.13}$$

Solving the ODEs (2.12) and (2.13) associated with the initial values, we obtain $r_0(\theta,\rho)$ and $r_1(\theta,\rho)$ as given in (2.7) and (2.8). Thus, the period of the periodic orbit passing through $(x,y)=(\rho,0)$ of system (1.5) can be computed as

$$\begin{aligned}T(\rho,\varepsilon)&=\int_0^{2\pi}\frac{1}{\dot\theta}\,d\theta=\int_0^{2\pi}\frac{1}{1+\varepsilon\sum_{i=1}^{m-1} r^i(\theta,\varepsilon;\rho)S_i(\theta)}\,d\theta\\&=2\pi-\left(\sum_{i=1}^{m-1}\int_0^{2\pi} r_0^i(\theta,\rho)S_i(\theta)\,d\theta\right)\varepsilon\\&\quad+\left(\int_0^{2\pi}\left\{\left(\sum_{i=1}^{m-1} r_0^i(\theta,\rho)S_i(\theta)\right)^2-\sum_{i=1}^{m-1} ir_0^{i-1}(\theta,\rho)r_1(\theta,\rho)S_i(\theta)\right\}d\theta\right)\varepsilon^2+O\left(\varepsilon^3\right).\end{aligned} \tag{2.14}$$

Therefore, from (2.7) and (2.14) we obtain the first two PBFs

$$\begin{aligned}T_1(\rho)&=-\sum_{i=1}^{m-1}\int_0^{2\pi} r_0^i(\theta,\rho)S_i(\theta)\,d\theta=-\sum_{i=1}^{m-1}\int_0^{2\pi}\frac{\rho^i S_i(\theta)}{(1+(1-n)\rho^{n-1}\int_0^\theta G_0(\alpha)\,d\alpha)^{\frac{i}{n-1}}}\,d\theta,\\T_2(\rho)&=\int_0^{2\pi}\left(\left(\sum_{i=1}^{m-1} r_0^i(\theta,\rho)S_i(\theta)\right)^2-\sum_{i=1}^{m-1} ir_0^{i-1}(\theta,\rho)r_1(\theta,\rho)S_i(\theta)\right)d\theta,\end{aligned}$$

i.e., (2.5) and (2.6) are proved.

When n is odd, it follows from (2.2) and (2.4) that $S_{2k+1}(\theta+\pi)=-S_{2k+1}(\theta)$ and

$$\int_0^{\theta+\pi} G_0(\alpha)\,d\alpha=\int_{-\pi}^{\theta} G_0(\alpha+\pi)\,d\alpha=\int_{-\pi}^{\theta} G_0(\alpha)\,d\alpha=\int_{-\pi}^{0} G_0(\alpha)\,d\alpha+\int_0^{\theta} G_0(\alpha)\,d\alpha=\int_0^{\theta} G_0(\alpha)\,d\alpha.$$

Then,

$$\begin{aligned}
&\int_0^{2\pi} \frac{\rho^{2k+1}S_{2k+1}(\theta)}{(1+(1-n)\rho^{n-1}\int_0^\theta G_0(\alpha)\,d\alpha)^{\frac{2k+1}{n-1}}}\,d\theta \\
&=\int_0^{\pi} \frac{\rho^{2k+1}S_{2k+1}(\theta)}{(1+(1-n)\rho^{n-1}\int_0^\theta G_0(\alpha)\,d\alpha)^{\frac{2k+1}{n-1}}}\,d\theta+\int_\pi^{2\pi} \frac{\rho^{2k+1}S_{2k+1}(\theta)}{(1+(1-n)\rho^{n-1}\int_0^\theta G_0(\alpha)\,d\alpha)^{\frac{2k+1}{n-1}}}\,d\theta \\
&=\int_0^{\pi} \frac{\rho^{2k+1}S_{2k+1}(\theta)}{(1+(1-n)\rho^{n-1}\int_0^\theta G_0(\alpha)\,d\alpha)^{\frac{2k+1}{n-1}}}\,d\theta+\int_0^{\pi} \frac{\rho^{2k+1}S_{2k+1}(\theta+\pi)}{(1+(1-n)\rho^{n-1}\int_0^{\theta+\pi} G_0(\alpha)\,d\alpha)^{\frac{2k+1}{n-1}}}\,d\theta \\
&=\int_0^{\pi} \frac{\rho^{2k+1}S_{2k+1}(\theta)}{(1+(1-n)\rho^{n-1}\int_0^\theta G_0(\alpha)\,d\alpha)^{\frac{2k+1}{n-1}}}\,d\theta+\int_0^{\pi} \frac{-\rho^{2k+1}S_{2k+1}(\theta)}{(1+(1-n)\rho^{n-1}\int_0^\theta G_0(\alpha)\,d\alpha)^{\frac{2k+1}{n-1}}}\,d\theta \\
&=0. \qquad (2.15)
\end{aligned}$$

Therefore, from (2.5) and (2.15) we obtain (2.9) as n is odd. □

Using the same method as for $r_1(\theta,\rho)$, we can calculate $r_i(\theta,\rho)$ for $i\geqslant 2$ because, similar to (2.13), a first order differential equation of $r_i(\theta,\rho)$ can be obtained and solved. Thus, T_i can be obtained similarly for $i\geqslant 3$ from the proof of Theorem 2.1.

It is a hard work to compute T_i's and find the critical periods with those T_i's for perturbed *nonlinear* isochronous centers. Actually, all T_i's in [11] are proved to be polynomials because the considered case is a perturbed linear isochronous center. However, T_i's are not necessary to be polynomials for a perturbed nonlinear isochronous center. For example, T_1 given in [15,16] for perturbed quadratic isochronous centers is not a polynomial. In our paper we extend the study of perturbed n-th order isochronous centers. There are more difficulties in computing T_i's. In addition to [15,16], we intend not only to give the number of critical periods but also to determine their location. For this purpose, we define

$$P(\rho):=\sum_{i=1}^{m-1} i\rho^{i-1}\int_0^{2\pi} \frac{S_i(\theta)}{(1+(1-n)\rho^{n-1}\int_0^\theta G_0(\alpha)\,d\alpha)^{\frac{i+n-1}{n-1}}}\,d\theta$$

and call it the *judging function*. For odd n, using the same method as in the proof of Theorem 2.1, we can simplify the judging function to write it as

$$P(\rho)=\sum_{k=0}^{[\frac{m-1}{2}]} 2k\rho^{2k-1}\int_0^{2\pi} \frac{S_{2k}(\theta)}{(1+(1-n)\rho^{n-1}\int_0^\theta G_0(\alpha)\,d\alpha)^{\frac{2k+n-1}{n-1}}}\,d\theta,$$

where $S_0(\theta)\equiv 0$ and each $S_i(\theta)$ with odd index i disappears.

Theorem 2.2. *Let the annulus of periodic orbits of system* $(1.5)|_{\varepsilon=0}$ *surrounding* O *be parameterized by* $\rho \in (0, K)$, *where* $K > 0$. *Then, for a sufficiently small* ε,

(i) *the period function* $T(\rho, \varepsilon)$ *of* (1.5) *is increasing (resp. decreasing) in* $\rho \in (0, K)$ *if* $P(\rho) < 0$ *(resp.* > 0*) for all* $\rho \in (0, K)$;
(ii) *if* $P(\rho) \not\equiv 0$, *then the number of critical periods of* $T(\rho, \varepsilon)$ *in* $(0, K)$ *is less than or equal to the number of zeros (counting with multiplicity) of* $P(\rho)$ *in* $(0, K)$. *Moreover, there is exactly one critical period of* (1.5) *whose value* ρ *tends to* ρ_* *as* ε *tends to zero if* $P(\rho)$ *has a simple zero* ρ_* *in* $(0, K)$.

Proof. From (1.3),

$$\frac{\partial T(\rho, \varepsilon)}{\partial \rho} = \sum_{i=1}^{+\infty} \frac{dT_i(\rho)}{d\rho} \varepsilon^i := \varepsilon \Gamma(\rho, \varepsilon),$$

where

$$\Gamma(\rho, \varepsilon) = \frac{dT_1(\rho)}{d\rho} + \sum_{i=1}^{+\infty} \frac{dT_{i+1}(\rho)}{d\rho} \varepsilon^i.$$

On the other hand, by Theorem 2.1

$$\begin{aligned}
\frac{dT_1(\rho)}{d\rho} &= -\sum_{i=1}^{m-1} \Bigg\{ i\rho^{i-1} \int_0^{2\pi} \frac{S_i(\theta)}{(1 + (1-n)\rho^{n-1} \int_0^\theta G_0(\alpha)\, d\alpha)^{\frac{i}{n-1}}}\, d\theta \\
&\quad + i\rho^i \int_0^{2\pi} \frac{(n-1)\rho^{n-2} S_i(\theta) \int_0^\theta G_0(\alpha)\, d\alpha}{(1 + (1-n)\rho^{n-1} \int_0^\theta G_0(\alpha)\, d\alpha)^{\frac{i+n-1}{n-1}}} d\theta \Bigg\} \\
&= -\sum_{i=1}^{m-1} i\rho^{i-1} \int_0^{2\pi} \frac{S_i(\theta)}{(1 + (1-n)\rho^{n-1} \int_0^\theta G_0(\alpha)\, d\alpha)^{\frac{i+n-1}{n-1}}}\, d\theta \\
&= -P(\rho).
\end{aligned} \tag{2.16}$$

Thus, for sufficiently small ε, $\partial T(\rho, \varepsilon)/\partial \rho$ has the opposite sign to the sign of function $P(\rho)$ if $P(\rho) \neq 0$, which immediately implies the correction of statement (i).

Assume that ρ_* is a zero of multiplicity m of $P(\rho)$ in $(0, K)$. In order to prove the first part of (ii), it suffices to prove that $T(\rho, \varepsilon)$ has at most m critical periods whose values tend to ρ_* as ε tends to 0. For a reduction to absurdity, suppose $T(\rho, \varepsilon)$ has at least $m + 1$ critical periods whose values tend to ρ_* as ε tends to 0, that is, $\partial T(\rho, \varepsilon)/\partial \rho$ has at least $m + 1$ zeros near ρ_*. By Rolle's Theorem, $\partial^2 T(\rho, \varepsilon)/\partial \rho^2$ has at least m zeros near ρ_*. Thus, $\partial^m T(\rho, \varepsilon)/\partial \rho^m$ has at least 2 zeros near ρ_*. On the other hand,

$$\begin{aligned}
&\frac{\partial^m T(\rho, \varepsilon)}{\partial \rho^m}\bigg|_{(\rho,\varepsilon)=(\rho_*,0)} \\
&= \left(\frac{d^m T_1(\rho)}{d\rho^m} + \sum_{i=1}^{+\infty} \frac{d^m T_i(\rho)}{d\rho^m} \varepsilon^i \right)\bigg|_{(\rho,\varepsilon)=(\rho_*,0)} = (1-n)\rho_*^{n-2} \frac{d^{m-1} P(\rho_*)}{d\rho^{m-1}} = 0,
\end{aligned}$$

$$\left.\frac{\partial^{m+1}T(\rho,\varepsilon)}{\partial\rho^{m+1}}\right|_{(\rho,\varepsilon)=(\rho_*,0)}$$
$$=\left.\left(\frac{d^{m+1}T_1(\rho)}{d\rho^{m+1}}+\sum_{i=1}^{+\infty}\frac{d^{m+1}T_i(\rho)}{d\rho^{m+1}}\varepsilon^i\right)\right|_{(\rho,\varepsilon)=(\rho_*,0)}=(1-n)\rho_*^{n-2}\frac{d^mP(\rho_*)}{d\rho^m}\neq 0.$$

By the Implicit Function Theorem, $\partial^mT(\rho,\varepsilon)/\partial\rho^m$ has a unique zero near ρ_*. Therefore, the contradiction implies that $T(\rho,\varepsilon)$ has at most m critical periods whose values tend to ρ_* as ε tends to 0. The first part of (ii) is proved.

Assume that $\rho_*\in(0,K)$ is a simple zero of $P(\rho)$. In order to prove the second part of (ii), it suffices to find a unique function $\rho(\varepsilon)$ such that $\Gamma(\rho(\varepsilon),\varepsilon)=0$ and $\rho(0)=\rho_*$. From (2.16) we get that $T_1'(\rho_*)=(1-n)\rho_*^{n-2}P(\rho_*)=0$ and, moreover, $T_1''(\rho_*)=-P'(\rho_*)\neq 0$ if $n=2$ and $T_1''(\rho_*)=(1-n)\rho_*^{n-2}P'(\rho_*)\neq 0$ if $n\geqslant 3$. Thus, ρ_* is a simple zero of $dT_1(\rho)/d\rho$. Therefore,

$$\Gamma(\rho_*,0)=\left.\frac{d}{d\rho}T_1(\rho)\right|_{\rho=\rho_*}=0,\qquad \left.\frac{\partial}{\partial\rho}\Gamma(\rho,\varepsilon)\right|_{(\rho,\varepsilon)=(\rho_*,0)}=\left.\frac{d^2}{d\rho^2}T_1(\rho)\right|_{\rho=\rho_*}\neq 0.$$

By the Implicit Function Theorem, there is a unique function $\rho(\varepsilon)$ such that $\Gamma(\rho(\varepsilon),\varepsilon)=0$ and $\rho(0)=\rho_*$. The proof of the second part of (ii) is completed. □

As shown in (2.16), $P(\rho)=-dT_1(\rho)/d\rho$. If $P(\rho)\equiv 0$ for all $\rho\in(0,K)$, then $T_1(\rho)\equiv C$ (C is a constant). That is, for sufficiently small ε, either $T_1(\rho)\equiv 0$ or $T(\rho,\varepsilon)$ is monotonic for ρ if $C\neq 0$. On the other hand, the case of $P(\rho)\not\equiv 0$ is discussed in (ii) of Theorem 2.2. As indicated in [16], if $T_1(\rho)\equiv\cdots\equiv T_{\ell-1}(\rho)\equiv 0$, $T_\ell(\rho)\not\equiv 0$ and the equation $dT_\ell(\rho)/d\rho=0$ for $\rho\in(0,K)$ has exactly j roots, all of them being simple, then by the Implicit Function Theorem the equation $\partial T(\rho,\varepsilon)/\partial\rho=0$ has exactly j solutions, which correspond to critical periods of the system. In this case we say that j critical periods *bifurcate* (up to the ℓ-th order in ε) from the annulus of periodic orbits surrounding the isochronous center. In this sense, the result (ii) in our Theorem 2.2 actually gives the number of bifurcated critical periods (up to the first order in ε) when $P(\rho)\not\equiv 0$.

3. Quadratic perturbed systems

In this section we apply Theorems 2.1 and 2.2 to quadratic systems (i.e., $n=m=2$). We not only judge whether there appear critical periods but also determine the location of those critical periods.

For $n=m=2$, system (1.5) becomes

$$\begin{cases}\dot{x}=-y+a_{11}xy+\varepsilon b_{11}xy,\\ \dot{y}=x+a_{11}y^2+\varepsilon\big(c_{20}x^2+c_{02}y^2\big),\end{cases}\tag{3.1}$$

where $a_{11}\neq 0$. O is a center of system (3.1) because of the symmetry. It is actually an isochronous center of system $(3.1)|_{\varepsilon=0}$ as shown in [3,7,20]. We need only to consider the case $a_{11}>0$ in (3.1); otherwise, one can apply the transformation $x\to -x$, $y\to -y$ to change (3.1) into the same form with opposite signs to the coefficients a_{11}, b_{11}, c_{20}, c_{02}. Further, by a rescaling

$$u=(a_{11}+\varepsilon b_{11})x,\qquad v=(a_{11}+\varepsilon b_{11})y,\tag{3.2}$$

system (3.1) can be simplified as

$$\begin{cases}\dot{u}=-v+uv,\\ \dot{v}=u+v^2+\varepsilon\big(C_{20}u^2+C_{02}v^2\big),\end{cases}\tag{3.3}$$

where

$$C_{20} = \frac{c_{20}}{a_{11} + \varepsilon b_{11}}, \qquad C_{02} = \frac{c_{02} - b_{11}}{a_{11} + \varepsilon b_{11}}. \tag{3.4}$$

We first consider system (3.3) for general real C_{20} and C_{02}. Clearly, system $(3.3)|_{\varepsilon=0}$ has a first integral $H(u,v) = (2u + v^2 - 1)/(1-u)^2$ and for each $\rho \in (0, 1/2)$ the orbit passing through $(\rho, 0)$ lies in the annulus of periodic orbits.

Theorem 3.1. *For system* (3.3) *with general real C_{20} and C_{02}, the first PBF*

$$T_1(\rho) = (C_{02} - 3C_{20})\pi + \frac{4(C_{20} - C_{02})\pi}{\rho} + \frac{2(C_{02} + C_{20})\pi}{\rho^2} + \frac{2(1-\rho)(C_{20}\rho^2 + 2(C_{02} - C_{20})\rho + C_{20} - C_{02})\pi}{\rho^2\sqrt{1-2\rho}},$$

where $\rho \in (0, 1/2)$. Furthermore, for sufficiently small ε,

(i) *center O preserves the isochronicity when $C_{20} = C_{02} = 0$; period function $T(\rho, \varepsilon)$ is increasing* (*resp. decreasing*) *for $\rho \in (0, 1/2)$ when $C_{20} = 0$ and $C_{02} > 0$* (*resp.* < 0);
(ii) *there is at most one critical period in* $(0, 1/2)$ *when $C_{20} \neq 0$. Moreover, there is exactly one critical period in* $(0, 1/2)$ *if and only if*

$$r_* := \frac{C_{02}}{C_{20}} < -3. \tag{3.5}$$

The value ρ of the unique critical period tends to the number

$$\rho_* := \sqrt{(1 - r_*)(1 - r_* - 2\sqrt{1 - r_*})} - \sqrt{1 - r_* - 2\sqrt{1 - r_*}} + 2\sqrt{1 - r_*} + r_* - 1 \tag{3.6}$$

as ε tends to 0*. Otherwise, $T(\rho, \varepsilon)$ is increasing* (*resp. decreasing*) *for $\rho \in (0, 1/2)$ if $C_{20} > 0$* (*resp.* < 0).

Proof. From (2.2), (2.3) and (2.4) we obtain for system (3.3) that

$$G_0(\theta) = \sin\theta, \qquad G_1(\theta) = C_{20}\cos^2\theta\sin\theta + C_{02}\sin^3\theta, \qquad S_1(\theta) = C_{20}\cos^3\theta + C_{02}\cos\theta\sin^2\theta.$$

From (2.5) given in Theorem 2.1 we obtain that

$$\begin{aligned} T_1(\rho) &= -\int_0^{2\pi} \frac{\rho(C_{20}\cos^3\theta + C_{02}\cos\theta\sin^2\theta)}{1 - \rho + \rho\cos\theta}\,d\theta \\ &= -\int_0^{2\pi} \left(1 - \frac{1-\rho}{1 - \rho + \rho\cos\theta}\right)\left(C_{20}\cos^2\theta + C_{02}\sin^2\theta\right)d\theta \\ &= (-C_{20} - C_{02})\pi + (1-\rho)\int_0^{2\pi} \frac{(C_{20} - C_{02})\cos^2\theta + C_{02}}{1 - \rho + \rho\cos\theta}\,d\theta \end{aligned}$$

$$
\begin{aligned}
&= (-C_{20} - C_{02})\pi + (1-\rho)\int_0^{2\pi} \frac{(1-\rho)(C_{02}-C_{20})}{\rho^2}\,d\theta \\
&\quad + \frac{(1-\rho)(C_{20}\rho^2 + 2(C_{02}-C_{20})\rho + C_{20} - C_{02})}{\rho^2}\int_0^{2\pi} \frac{1}{1-\rho+\rho\cos\theta}\,d\theta \\
&= (C_{02} - 3C_{20})\pi + \frac{4(C_{20}-C_{02})\pi}{\rho} + \frac{2(C_{02}+C_{20})\pi}{\rho^2} \\
&\quad + \frac{2(1-\rho)(C_{20}\rho^2 + 2(C_{02}-C_{20})\rho + C_{20} - C_{02})\pi}{\rho^2\sqrt{1-2\rho}}.
\end{aligned}
$$

From the expression of $T_1(\rho)$, it is easy to compute

$$
\frac{d}{d\rho}T_1(\rho) = \frac{(1-\delta)\pi F(\delta)}{\delta^3(\delta+1)^3},
$$

where $\delta = \sqrt{1-2\rho}$ and

$$
F(\delta) = C_{20}\delta^4 + 4C_{20}\delta^3 + (2C_{20} + 4C_{02})\delta^2 + 4C_{20}\delta + C_{20}. \tag{3.7}
$$

Another way is to use Theorem 2.2 and compute $P(\rho)$. By formula (2.558) in [17, p. 182] we calculate

$$
P(\rho) = \frac{(\delta-1)\pi F(\delta)}{\delta^3(\delta+1)^3}. \tag{3.8}
$$

Obviously, $\delta \in (0,1)$. Note that $F(\delta) = 4C_{02}\delta^2$ as $C_{20} = 0$. It implies that $dT_1(\rho)/d\rho < 0$ (resp. > 0) if $C_{02} > 0$ (resp. < 0). Therefore, for sufficiently small ε, $\partial T(\rho,\varepsilon)/\partial\rho < 0$ (resp. > 0) for $\rho \in (0,1/2)$ if $C_{02} > 0$ (resp. < 0). For $C_{20} = C_{02} = 0$, system (3.3) satisfies that $\dot\theta \equiv 1$ in the polar coordinates, which means that the origin is an isochronous center. The proof of (i) is completed.

Assume that $C_{20} \neq 0$. By (3.7) we obtain four zeros of $F(\delta)$ as follows:

$$
\begin{aligned}
\delta_\pm &= -1 - \sqrt{1-r_*} \pm \sqrt{1 - r_* - 2\sqrt{1-r_*}}, \\
\tilde\delta_\pm &= -1 + \sqrt{1-r_*} \pm \sqrt{1 - r_* - 2\sqrt{1-r_*}},
\end{aligned} \tag{3.9}
$$

where r_* is given in (3.5). It is not difficult to check that $\delta_\pm \notin (0,1)$ and $\tilde\delta_+ \notin (0,1)$. Moreover, $\tilde\delta_- \in (0,1)$ if and only if $r_* < -3$. Thus, there is at most one critical period in $(0,1/2)$. When $r_* \geqslant -3$, $P(\rho) < 0$ (resp. > 0) as $C_{20} > 0$ (resp. < 0) for $\rho \in (0,1/2)$ because $F(\delta)$ has no zeros in $(0,1)$. On the other hand, $\rho = (1-\tilde\delta_-^2)/2 = \rho_*$, where ρ_* is given in (3.6). When $r_* < -3$, we know that $\tilde\delta_-$ is a simple zero of $F(\delta)$ and

$$
\frac{d}{d\rho}P(\rho)\bigg|_{\rho=\rho_*} = \frac{d}{d\delta}\frac{(\delta-1)\pi F(\delta)}{\delta^3(\delta+1)^3}\bigg|_{\delta=\tilde\delta_-} \cdot \frac{d\delta}{d\rho}\bigg|_{\rho=\rho_*} = \frac{(\tilde\delta_- - 1)\pi F'(\tilde\delta_-)}{\tilde\delta_-^3(\tilde\delta_- + 1)^3} \cdot \frac{-1}{\sqrt{1-2\rho_*}} \neq 0,
$$

that is, ρ_* is a simple zero of $P(\rho)$. The proof of result (ii) is completed by Theorem 2.2. □

From (3.7) and (3.8) we know that $P(\rho) \equiv 0$ if and only if $C_{20} = C_{02} = 0$. In such a case the isochronicity is proved in Theorem 3.1. Thus, for system (3.3) we need only to consider the case

$P(\rho) \not\equiv 0$. As remarked at the end of Section 2, the bifurcation of critical periods of system (3.3) only depends on the first order of ε.

The bifurcation of critical periods for system (3.3) is also investigated in [15], where it is proved that, up to the first order in ε, at most one critical period bifurcates and that the sufficient and necessary condition for the system to have exactly one critical period is that $-1/3 < C_{20}/C_{02} < 0$. It is easy to see that these results are consistent with ours in Theorem 3.1. Moreover, from Theorem 3.1 we know the location of the unique critical period, which lies near

$$\tilde{\rho} := \sqrt{\left(1-\frac{C_{02}}{C_{20}}\right)\left(1-\frac{C_{02}}{C_{20}}-2\sqrt{1-\frac{C_{02}}{C_{20}}}\right)} - \sqrt{1-\frac{C_{02}}{C_{20}}-2\sqrt{1-\frac{C_{02}}{C_{20}}}} + 2\sqrt{1-\frac{C_{02}}{C_{20}}}+\frac{C_{02}}{C_{20}}-1.$$

The location of critical periods is important and helpful in the study of subharmonic bifurcations. For example, for the system considered above we know from Theorem 3.1 that for any small $0 < \xi \ll 1$ there exists $\varepsilon_0 > 0$ such that $T(\rho, \varepsilon)$ is monotonic for $\rho \in (0, \tilde{\rho} - \xi)$ when $\varepsilon < \varepsilon_0$, which is usually needed in the research of subharmonic bifurcations [18].

By Theorem 3.1 we obtain the result for system (3.1) with $a_{11} > 0$ as follows.

Corollary 3.1. *For system* (3.1) *with $a_{11} > 0$, the first PBF*

$$T_1(\rho) = \frac{(c_{02}-b_{11}-3c_{20})\pi}{a_{11}} + \frac{4(c_{20}+b_{11}-c_{02})\pi}{a_{11}^2\rho} + \frac{2(c_{02}-b_{11}+c_{20})\pi}{a_{11}^3\rho^2} + \frac{2(1-a_{11}\rho)(c_{20}a_{11}^2\rho^2+2(c_{02}-b_{11}-c_{20})a_{11}\rho+c_{20}+b_{11}-c_{02})\pi}{a_{11}^3\rho^2\sqrt{1-2a_{11}\rho}},$$

where $\rho \in (0, 1/(2a_{11}))$. Furthermore, for sufficiently small ε,

(i) *center O preserves the isochronicity when $c_{20} = 0$ and $b_{11} = c_{02}$; period function $T(\rho, \varepsilon)$ is increasing* (*resp. decreasing*) *for $\rho \in (0, 1/(2a_{11}))$ when $c_{20} = 0$ and $c_{02} - b_{11} > 0$* (*resp.* < 0);
(ii) *there is at most one critical period in $(0, 1/(2a_{11}))$ when $c_{20} \neq 0$. Moreover, there is exactly one critical period in $(0, 1/(2a_{11}))$ if and only if*

$$r_* := \frac{c_{02}-b_{11}}{c_{20}} < -3. \tag{3.10}$$

The value ρ of the unique critical period tends to the number

$$\rho_* := \frac{1}{a_{11}}\left(\sqrt{(1-r_*)(1-r_*-2\sqrt{1-r_*})} - \sqrt{1-r_*-2\sqrt{1-r_*}}+2\sqrt{1-r_*}+r_*-1\right) \tag{3.11}$$

as ε tends to 0. *Otherwise, period function $T(\rho, \varepsilon)$ is increasing* (*resp. decreasing*) *for $\rho \in (0, 1/(2a_{11}))$ if $c_{20} > 0$* (*resp.* < 0).

Proof. We need to apply Theorem 3.1 to system (3.3) associated with (3.4). With the restriction (3.4), the first PBF, given in Theorem 3.1, can be expressed as

$$\begin{aligned}\tilde{T}_1(\rho,\varepsilon)=&\frac{(c_{02}-b_{11}-3c_{20})\pi}{a_{11}+\varepsilon b_{11}}+\frac{4(c_{20}+b_{11}-c_{02})\pi}{(a_{11}+\varepsilon b_{11})\rho}+\frac{2(c_{02}-b_{11}+c_{20})\pi}{(a_{11}+\varepsilon b_{11})\rho^2}\\&+\frac{2(1-\rho)(c_{20}\rho^2+2(c_{02}-b_{11}-c_{20})\rho+c_{20}+b_{11}-c_{02})\pi}{(a_{11}+\varepsilon b_{11})\rho^2\sqrt{1-2\rho}},\end{aligned}$$

which depends on ε clearly. Note that the rescaling (3.2), which changes system (3.1) into system (3.3) with (3.4), gives the correspondence $(\rho,0)\mapsto((a_{11}+\varepsilon b_{11})\rho,0)$ from the (x,y)-space to the (u,v)-space. Thus, for system (3.1) with $a_{11}>0$ we obtain the first PBF

$$\begin{aligned}T_1(\rho)&=\tilde{T}_1\big((a_{11}+\varepsilon b_{11})\rho,\varepsilon\big)\big|_{\varepsilon=0}\\&=\frac{(c_{02}-b_{11}-3c_{20})\pi}{a_{11}}+\frac{4(c_{20}+b_{11}-c_{02})\pi}{a_{11}^2\rho}+\frac{2(c_{02}-b_{11}+c_{20})\pi}{a_{11}^3\rho^2}\\&\quad+\frac{2(1-a_{11}\rho)(c_{20}a_{11}^2\rho^2+2(c_{02}-b_{11}-c_{20})a_{11}\rho+c_{20}+b_{11}-c_{02})\pi}{a_{11}^3\rho^2\sqrt{1-2a_{11}\rho}}.\end{aligned}$$

Other results in the corollary can also be given from results in Theorem 3.1 similarly. □

4. Cubic perturbed systems

In this section we apply Theorems 2.1 and 2.2 to the case $n=m=3$. We give conditions for the existence of critical periods. Although the conditions are not so explicit as the condition (3.10) of Corollary 3.1 for quadratic systems, it is very convenient to judge whether the period function $T(\rho,\varepsilon)$ has some critical periods in a given range once the values of parameters are given. Moreover, the location can be found accurately and the number of critical periods is at most two in the considered region.

When $n=m=3$, system (1.5) is of the form

$$\begin{cases}\dot{x}=-y+a_{21}x^2y+\varepsilon\big(b_{11}xy+b_{21}x^2y+b_{03}y^3\big),\\ \dot{y}=x+a_{21}xy^2+\varepsilon\big(c_{20}x^2+c_{02}y^2+c_{30}x^3+c_{12}xy^2\big),\end{cases}\tag{4.1}$$

where $a_{21}\neq 0$. O is a center of system (4.1) because of the symmetry. Actually, it is an isochronous center of system $(4.1)|_{\varepsilon=0}$ as shown in [3,22,24]. Further, in the case $a_{21}>0$ system (4.1) can be simplified as

$$\begin{cases}\dot{u}=-v+au^2v+\varepsilon\big(B_{11}uv+B_{21}u^2v+B_{03}v^3\big),\\ \dot{v}=u+auv^2+\varepsilon\big(C_{20}u^2+C_{02}v^2+C_{30}u^3+C_{12}uv^2\big)\end{cases}\tag{4.2}$$

by the rescaling

$$u=\sqrt{a_{21}}x,\qquad v=\sqrt{a_{21}}y,\tag{4.3}$$

where $a=1$ and

$$\begin{gathered}B_{11}=\frac{b_{11}}{\sqrt{a_{21}}},\qquad B_{21}=\frac{b_{21}}{a_{21}},\qquad B_{03}=\frac{b_{03}}{a_{21}},\\ C_{20}=\frac{c_{20}}{\sqrt{a_{21}}},\qquad C_{02}=\frac{c_{02}}{\sqrt{a_{21}}},\qquad C_{30}=\frac{c_{30}}{a_{21}},\qquad C_{12}=\frac{c_{12}}{a_{21}}.\end{gathered}\tag{4.4}$$

Similarly, in the case $a_{21} < 0$ system (4.1) can be simplified as (4.2) by the rescaling $u = \sqrt{-a_{21}}x$, $v = \sqrt{-a_{21}}y$, where $a = -1$ and

$$B_{11} = \frac{b_{11}}{\sqrt{-a_{21}}}, \qquad B_{21} = -\frac{b_{21}}{a_{21}}, \qquad B_{03} = -\frac{b_{03}}{a_{21}},$$
$$C_{20} = \frac{c_{20}}{\sqrt{-a_{21}}}, \qquad C_{02} = \frac{c_{02}}{\sqrt{-a_{21}}}, \qquad C_{30} = -\frac{c_{30}}{a_{21}}, \qquad C_{12} = -\frac{c_{12}}{a_{21}}. \tag{4.5}$$

We first consider system (4.2) with $a = \pm 1$ for general real B_{ij}'s and C_{ij}'s. For $\varepsilon = 0$ the system has a first integral $H(u, v) = (1 + av^2)/(1 - au^2)$ and for $\rho \in (0, 1)$ (resp. $\rho > 0$) the orbit passing through $(\rho, 0)$ lies in the annulus of period orbits when $a = 1$ (resp. $a = -1$).

Theorem 4.1. *For system* (4.2) *with $a = 1$* (*resp. $a = -1$*) *and general real B_{ij}'s and C_{ij}'s, the first PBF*

$$T_1(\rho) = \frac{(1 - \sqrt{1 - \rho^2})(2C_{30}\rho^2 + (B_{21} + B_{03} - C_{30} - C_{12})\sqrt{1 - \rho^2} + 2B_{03} - 2C_{30})\pi}{(1 - \rho^2 + \sqrt{1 - \rho^2})},$$

where $\rho \in (0, 1)$ (*resp. $\rho \in (0, K_1)$, an interval corresponding to a region filled with period orbits*)*. Furthermore, for sufficiently small ε we have the following results*:

(i) *Center O preserves the isochronicity when*

$$C_{20} = B_{11} - C_{02} = 0, \tag{4.6}$$
$$C_{30} = B_{03} = B_{21} - C_{12} = 0. \tag{4.7}$$

The period function $T(\rho, \varepsilon)$ is increasing for $\rho \in (0, 1)$ (*resp. $\rho \in (0, K_1)$*) *when* (4.7) *does not hold and $G(\delta) < 0$ for $\delta \in (0, 1)$* (*resp. $\delta \in (1, \sqrt{1 + K_1^2})$*) *but decreasing for $\rho \in (0, 1)$* (*resp. $\rho \in (0, K_1)$*) *when* (4.7) *does not hold and $G(\delta) > 0$ for $\delta \in (0, 1)$* (*resp. $\delta \in (1, \sqrt{1 + K_1^2})$*)*, where*

$$G(\delta) := C_{30}\delta^4 + 2C_{30}\delta^3 + (C_{12} - B_{21})\delta^2 - 2B_{03}\delta - B_{03}. \tag{4.8}$$

(ii) *When* (4.7) *does not hold but $G(\delta)$ has a simple zero δ_* in $(0, 1)$* (*resp. in $(1, \sqrt{1 + K_1^2})$*)*, there is exactly one critical period tending to the number $\rho_* := \sqrt{1 - \delta_*^2}$* (*resp. $\rho_* := \sqrt{\delta_*^2 - 1}$*) *in $(0, 1)$* (*resp. in $(0, K_1)$*) *as ε tends to* 0.

(iii) *There are at most two critical periods in $(0, 1)$* (*resp. in $(0, K_1)$*) *and the maximum is achievable.*

Proof. We first consider the case $a = 1$. When both (4.6) and (4.7) hold, we find that $\dot{\theta} \equiv 1$ in (2.1) for system (4.2), which means that O is an isochronous center. From (2.2), (2.3) and (2.4) we obtain that for system (4.2)

$$G_0(\theta) = a\cos\theta\sin\theta, \qquad G_1(\theta) = (B_{11} + C_{20})\cos^2\theta\sin\theta + C_{02}\sin^3\theta,$$
$$S_1(\theta) = C_{20}\cos^3\theta + (C_{02} - B_{11})\cos\theta\sin^2\theta,$$
$$G_2(\theta) = (B_{21} + C_{30})\cos^3\theta\sin\theta + (B_{03} + C_{12})\cos\theta\sin^3\theta,$$
$$S_2(\theta) = C_{30}\cos^4\theta + (C_{12} - B_{21})\cos^2\theta\sin^2\theta - B_{03}\sin^4\theta. \tag{4.9}$$

From (2.5) given in Theorem 2.1 we obtain that

$$
\begin{aligned}
T_1(\rho) &= \int_0^{2\pi} \frac{\rho^2(B_{03}\sin^4\theta + (B_{21}-C_{12})\cos^2\theta\sin^2\theta - C_{30}\cos^4\theta)}{1-\rho^2\sin^2\theta}\,d\theta \\
&= \rho^2 B_{03}\int_0^{2\pi}\frac{\sin^4\theta}{1-\rho^2\sin^2\theta}\,d\theta + \rho^2(B_{21}-C_{12})\int_0^{2\pi}\frac{\cos^2\theta\sin^2\theta}{1-\rho^2\sin^2\theta}\,d\theta \\
&\quad - \rho^2 C_{30}\int_0^{2\pi}\frac{\cos^4\theta}{1-\rho^2\sin^2\theta}\,d\theta \\
&= \rho^2(B_{03}+C_{12}-B_{21}-C_{30})\int_0^{2\pi}\frac{\sin^4\theta}{1-\rho^2\sin^2\theta}\,d\theta \\
&\quad + \rho^2(B_{21}-C_{12}+3C_{30})\int_0^{2\pi}\frac{\sin^2\theta}{1-\rho^2\sin^2\theta}\,d\theta - \rho^2 C_{30}\int_0^{2\pi}\frac{1}{1-\rho^2\sin^2\theta}\,d\theta.
\end{aligned} \tag{4.10}
$$

By [17, p. 185], for $\rho \in (0,1)$ it is not difficult to compute

$$
\begin{aligned}
&\int_0^{2\pi}\frac{\sin^4\theta}{1-\rho^2\sin^2\theta}\,d\theta = \frac{-\pi}{\rho^2} + \frac{-2\pi}{\rho^4} + \frac{1}{\rho^4}\int_0^{2\pi}\frac{1}{1-\rho^2\sin^2\theta}\,d\theta, \\
&\int_0^{2\pi}\frac{\sin^2\theta}{1-\rho^2\sin^2\theta}\,d\theta = \frac{-2\pi}{\rho^2} + \frac{1}{\rho^2}\int_0^{2\pi}\frac{1}{1-\rho^2\sin^2\theta}\,d\theta, \\
&\int_0^{2\pi}\frac{1}{1-\rho^2\sin^2\theta}\,d\theta = \frac{2\pi}{\sqrt{1-\rho^2}}.
\end{aligned} \tag{4.11}
$$

It follows from (4.10) and (4.11) that

$$
T_1(\rho) = \frac{(1-\sqrt{1-\rho^2})(2C_{30}\rho^2 + (B_{21}+B_{03}-C_{30}-C_{12})\sqrt{1-\rho^2} + 2B_{03} - 2C_{30})\pi}{(1-\rho^2+\sqrt{1-\rho^2})}.
$$

Then, we calculate

$$
\frac{d}{d\rho}T_1(\rho) = \frac{-2\pi\, G(\delta)}{\delta^3(\delta+1)^2}\sqrt{1-\delta^2},
$$

where $G(\delta)$ is given in (4.8) and $\delta = \sqrt{1-\rho^2}$. By Theorem 2.2

$$
P(\rho) = \frac{\pi\, G(\delta)}{\delta^3(\delta+1)^2}. \tag{4.12}
$$

Therefore, the results in (i) can be obtained directly by (i) of Theorem 2.2.

From (4.12),

$$\frac{d}{d\rho}P(\rho) = \frac{d}{d\delta}\frac{\pi G(\delta)}{\delta^3(\delta+1)^2}\cdot\frac{d\delta}{d\rho} = \frac{\pi\delta(\delta+1)G'(\delta)-\pi(5\delta+3)G(\delta)}{\delta^4(\delta+1)^3}\cdot\frac{-\rho}{\delta}. \tag{4.13}$$

Obviously, it follows from (4.12) and (4.13) that $P(\rho)$ has a simple zero ρ_* in $(0,1)$ if and only if $G(\delta)$ has a simple zero $\delta_* := \sqrt{1-\rho_*^2}$ in $(0,1)$. Therefore, the results in (ii) follow from (ii) of Theorem 2.2.

If $C_{30}=0$, then $G(\delta)=(C_{12}-B_{21})\delta^2-2B_{03}\delta-B_{03}$. It is easy to see that $G(\delta)$ has at most one zero in $(0,1)$. Thus, by (4.12), $P(\rho)$ has at most one zero in $(0,1)$. Therefore, $T(\rho,\varepsilon)$ has at most one critical period in $(0,1)$ when $C_{30}=0$. For $C_{30}\neq 0$, assume that δ_1, δ_2, δ_3 and δ_4 are zeros of $G(\delta)$ and $\delta_i\in(0,1)$, $i=1,2,3$. Then, $\delta_1+\cdots+\delta_4+2=0$, which implies that $\delta_4<-2$. On the other hand, we get

$$\begin{aligned}
-\frac{B_{03}}{C_{30}} &= \delta_1\delta_2\delta_3\delta_4 < 0,\\
-\frac{B_{03}}{C_{30}} &= -\frac{1}{2}(\delta_1\delta_2\delta_3+\delta_1\delta_2\delta_4+\delta_1\delta_3\delta_4+\delta_2\delta_3\delta_4) > -\frac{1}{2}\delta_1\delta_2(\delta_3+\delta_4) > 0.
\end{aligned}$$

This contradiction means that $G(\delta)$ has at most two zeros in $(0,1)$, implying that $P(\rho)$ has at most two zeros in $(0,1)$. Thus, $T(\rho,\varepsilon)$ has at most two critical periods in $(0,1)$ when $C_{30}\neq 0$. In order to show that the maximum is achievable, we construct an example by choosing B_{21}, B_{03}, C_{30} and C_{12} in system (4.2) such that

$$64(C_{12}-B_{21})+197C_{30}=0, \qquad 512B_{03}+117C_{30}=0,$$

where $C_{30}\neq 0$. Then, by (4.8),

$$G(\delta)=C_{30}\left(\delta^4+2\delta^3-\frac{197}{64}\delta^2+\frac{117}{256}\delta+\frac{117}{512}\right)=\frac{C_{30}(2\delta-1)(4\delta-3)(64\delta^2+208\delta+39)}{512}.$$

Thus, $G(\delta)$ has two simple zeros $1/2$ and $3/4$ in $(0,1)$. Therefore, $T(\rho,\varepsilon)$ has exactly two critical periods in $(0,1)$. The values of ρ of these two critical periods tend to $\sqrt{3}/2$ and $\sqrt{7}/4$, respectively, as ε tends to 0.

The proof for the case $a=-1$ is similar to the case $a=1$ and, hence, omitted. We only show that the maximum number 2 is achievable. Take B_{21}, B_{03}, C_{30} and C_{12} such that

$$\begin{aligned}
16\big(3\tilde{K}_1^2+9\tilde{K}_1+4\big)(C_{12}-B_{21})+3\big(19\tilde{K}_1^4+109\tilde{K}_1^3+207\tilde{K}_1^2+143\tilde{K}_1+34\big)C_{30}=0,\\
64\big(3\tilde{K}_1^2+9\tilde{K}_1+4\big)B_{03}+\big(45\tilde{K}_1^5+219\tilde{K}_1^4+374\tilde{K}_1^3+282\tilde{K}_1^2+93\tilde{K}_1+11\big)C_{30}=0,
\end{aligned}$$

where $C_{30}\neq 0$ and $\tilde{K}_1=\sqrt{1+K_1^2}$. Then, by (4.8),

$$\begin{aligned}
G(\delta) &= C_{30}\left(\delta^4+2\delta^3-\frac{3(19\tilde{K}_1^4+109\tilde{K}_1^3+207\tilde{K}_1^2+143\tilde{K}_1+34)}{16(3\tilde{K}_1^2+9\tilde{K}_1+4)}\delta^2\right.\\
&\quad\left.+\frac{45\tilde{K}_1^5+219\tilde{K}_1^4+374\tilde{K}_1^3+282\tilde{K}_1^2+93\tilde{K}_1+11}{64(3\tilde{K}_1^2+9\tilde{K}_1+4)}(2\delta+1)\right)\\
&= \big\{\big(24\tilde{K}_1^2+72\tilde{K}_1+32\big)\delta^2+\big(30\tilde{K}_1^3+156\tilde{K}_1^2+238\tilde{K}_1+88\big)\delta+15\tilde{K}_1^3+53\tilde{K}_1^2+49\tilde{K}_1+11\big\}\\
&\quad\cdot\frac{C_{30}(2\delta-1-\tilde{K}_1)(4\delta-1-3\tilde{K}_1)}{64(3\tilde{K}_1^2+9\tilde{K}_1+4)}.
\end{aligned}$$

Thus, $G(\delta)$ has exactly two simple zeros $(1+\tilde{K}_1)/2$ and $(1+3\tilde{K}_1)/4$ in $(1, \sqrt{1+K_1^2})$. Therefore, $T(\rho, \varepsilon)$ has exactly two critical periods in $(0, K_1)$. The values of ρ of these two critical periods tend to $\sqrt{\tilde{K}_1^2+2\tilde{K}_1-3}/2$ and $\sqrt{9\tilde{K}_1^2+6\tilde{K}_1-15}/4$, respectively, as ε tends to 0. $\square$

Actually, in Theorem 4.1 the reason why we change $(0,1)$ for the case $a=1$ into $(0, K_1)$ for the case $a=-1$ is that for system (4.2) with $a=-1$ we need equalities

$$\begin{aligned}
&\int_0^{2\pi} \frac{\sin^4\theta}{1+\rho^2\sin^2\theta}\,d\theta = \frac{\pi}{\rho^2} + \frac{-2\pi}{\rho^4} + \frac{1}{\rho^4}\int_0^{2\pi} \frac{1}{1+\rho^2\sin^2\theta}\,d\theta,\\
&\int_0^{2\pi} \frac{\sin^2\theta}{1+\rho^2\sin^2\theta}\,d\theta = \frac{2\pi}{\rho^2} + \frac{-1}{\rho^2}\int_0^{2\pi} \frac{1}{1+\rho^2\sin^2\theta}\,d\theta,\\
&\int_0^{2\pi} \frac{1}{1+\rho^2\sin^2\theta}\,d\theta = \frac{2\pi}{\sqrt{1+\rho^2}},
\end{aligned} \tag{4.14}$$

which correspond to the equalities in (4.11) for the case $a=1$. Moreover, the equalities in (4.14) hold for any positive ρ but the equalities in (4.11) hold only for positive ρ less than 1. Consequently, it is necessary to consider zeros of $G(\delta)$ in $(1, \sqrt{1+K_1^2})$ for the case $a=-1$ but in $(0,1)$ for the case $a=1$.

From Theorem 4.1 we obtain the result for system (4.1) as follows.

Corollary 4.1. *For system* (4.1) *with* $a_{21}>0$ (*resp.* $a_{21}<0$), *the first PBF*

$$T_1(\rho) = \frac{(1-\sqrt{1-a_{21}\rho^2})(2c_{30}a_{21}\rho^2 + (b_{21}+b_{03}-c_{30}-c_{12})\sqrt{1-a_{21}\rho^2} + 2b_{03} - 2c_{30})\pi}{a_{21}(1-a_{21}\rho^2+\sqrt{1-a_{21}\rho^2})},$$

where $\rho \in (0, 1/\sqrt{a_{21}})$ (*resp.* $\rho \in (0, K_2)$, *an interval corresponding to a region filled with period orbits*). *Furthermore, for sufficiently small* ε *we have the following results*:

(i) *Center* O *preserves the isochronicity when*

$$c_{20} = b_{11} - c_{02} = 0, \tag{4.15}$$

$$c_{30} = b_{03} = b_{21} - c_{12} = 0. \tag{4.16}$$

The period function $T(\rho,\varepsilon)$ *is increasing for* $\rho \in (0, 1/\sqrt{a_{21}})$ (*resp.* $\rho \in (0, K_2)$) *when* (4.16) *does not hold and* $G(\delta)<0$ *for* $\delta \in (0,1)$ (*resp.* $\delta \in (1, \sqrt{1-a_{21}K_2^2})$) *but decreasing for* $\rho \in (0, 1/\sqrt{a_{21}})$ (*resp.* $\rho \in (0, K_2)$) *when* (4.16) *does not hold and* $G(\delta)>0$ *for* $\delta \in (0,1)$ (*resp.* $\delta \in (1, \sqrt{1-a_{21}K_2^2})$), *where*

$$G(\delta) := c_{30}\delta^4 + 2c_{30}\delta^3 + (c_{12}-b_{21})\delta^2 - 2b_{03}\delta - b_{03}. \tag{4.17}$$

(ii) *When* (4.16) *does not hold but* $G(\delta)$ *has a simple zero* δ_* *in* $(0,1)$ (*resp. in* $(1, \sqrt{1-a_{21}K_2^2})$), *there is exactly one critical period tending to the number* $\rho_* := \sqrt{(1-\delta_*^2)/a_{21}}$ *in* $(0, 1/\sqrt{a_{21}})$ (*resp. in* $(0, K_2)$) *as* ε *tends to* 0.

(iii) *There are at most two critical periods in* $(0, 1/\sqrt{a_{21}})$ (*resp. in* $(0, K_2)$) *and the maximum is achievable.*

Proof. We only give the proof for the case $a_{21} > 0$. For the case $a_{21} < 0$ the proof is similar and, hence, omitted. Consider $a_{21} > 0$. We need to apply Theorem 4.1 to system (4.2) associated with (4.4). With the restriction (4.4), the first PBF, given in Theorem 4.1, can be expressed as

$$\tilde{T}_1(\rho) = \frac{(1-\sqrt{1-\rho^2})(2c_{30}\rho^2 + (b_{21}+b_{03}-c_{30}-c_{12})\sqrt{1-\rho^2} + 2b_{03} - 2b_{30})\pi}{a_{21}(1-\rho^2+\sqrt{1-\rho^2})}.$$

Note that the rescaling (4.3), which changes system (4.1) into system (4.2) with $a = 1$ and (4.4), gives the correspondence $(\rho, 0) \mapsto (\sqrt{a_{21}}\rho, 0)$ from the (x, y)-space to the (u, v)-space. Thus, for system (4.1) with $a_{21} > 0$ we obtain the first PBF

$$\begin{aligned} T_1(\rho) &= \tilde{T}_1(\sqrt{a_{21}}\rho) \\ &= \frac{(1-\sqrt{1-a_{21}\rho^2})(2c_{30}a_{21}\rho^2 + (b_{21}+b_{03}-c_{30}-c_{12})\sqrt{1-a_{21}\rho^2} + 2b_{03} - 2b_{30})\pi}{a_{21}(1-a_{21}\rho^2+\sqrt{1-a_{21}\rho^2})}. \end{aligned}$$

Other results in the corollary can be given from results in Theorem 4.1 similarly. □

In Corollary 4.1 we are not able to give the location of critical periods for system (4.1) explicitly, although it is given for system (3.1) in Corollary 3.1. However, one can easily find the location by computing zeros of $G(\delta)$ as soon as the bifurcation coefficients of system (4.1) are given. For example, fix

$$a_{21} = 4, \qquad b_{21} = 2, \qquad b_{03} = -\frac{2448}{7375}, \qquad c_{30} = 1, \qquad c_{12} = -\frac{2343}{1475}$$

in system (4.1). Then

$$G(\delta) = \frac{(5\delta-3)(5\delta-4)(295\delta^2 + 1003\delta + 204)}{7375},$$

which has two zeros $\delta_1 = 3/5$ and $\delta_2 = 4/5$ in $(0, 1)$. Thus, by Corollary 4.1 there are exactly two critical periods in $(0, 1/2)$ near $2/5$ and $3/10$, respectively.

In Corollary 4.1 we give the expression of $T_1(\rho)$ and the results for critical period bifurcation for system (4.1) when either (4.16) does not hold or both (4.15) and (4.16) hold. In the remainder of this section we consider the case when (4.16) holds but (4.15) does not hold. By the expression of $T_1(\rho)$ given in Corollary 4.1 it is easy to check that $T_1(\rho) \equiv 0$ in such a case, which means that we need to consider the bifurcation of critical periods up to higher (at least second) order in ε for system (4.1). From (1.3) we obtain

$$\frac{\partial T(\rho,\varepsilon)}{\partial \rho} = \sum_{i=1}^{+\infty} \frac{dT_i(\rho)}{d\rho}\varepsilon^i = \varepsilon^2\left(\frac{dT_2(\rho)}{d\rho} + \sum_{i=1}^{+\infty} \frac{dT_{i+2}(\rho)}{d\rho}\varepsilon^i\right),$$

which implies that the bifurcation of critical periods depends on $dT_2(\rho)/d\rho$. Thus, it is necessary to compute $T_2(\rho)$. Similarly to the computation of $T_1(\rho)$, we first consider the general system (4.2) with $a = \pm 1$, i.e., without condition (4.4) or (4.5).

Theorem 4.2. *Assume that* (4.7) *holds but* (4.6) *does not hold for system* (4.2) *with* $a = \pm 1$ *and general real* B_{ij}*'s and* C_{ij}*'s. Then the first PBF* $T_1(\rho) \equiv 0$ *and the second one*

$$T_2(\rho) = \frac{-\pi}{4a\rho^4} g_1(\rho) + \frac{\pi}{a\rho^4\sqrt{1-a\rho^2}} g_2(\rho)$$
$$+ \begin{cases} \frac{\Sigma(\rho)}{\rho} \int_0^\pi \frac{S_1(\theta)\ln(\sqrt{1-\rho^2\sin^2\theta}+\rho\cos\theta)}{(1-\rho^2\sin^2\theta)^{3/2}}\, d\theta, & \rho \in (0,1) \text{ for } a=1, \\ \frac{\Sigma(\rho)}{-\rho} \int_0^\pi \frac{S_1(\theta)\arcsin(\frac{\rho\cos\theta}{\sqrt{1+\rho^2}})}{(1+\rho^2\sin^2\theta)^{3/2}}\, d\theta, & \rho \in (0,K_1) \text{ for } a=-1, \end{cases} \tag{4.18}$$

where $S_1(\theta) = C_{20}\cos^3\theta + (C_{02} - B_{11})\cos\theta\sin^2\theta$, $g_1(\rho) = (-B_{11}^2 + 4B_{11}C_{02} - 3C_{02}^2 + 4B_{11}C_{20} + 2C_{02}C_{20} + 9C_{20}^2)\rho^4 - 2(B_{11} - C_{02} + C_{20})(B_{11} + C_{02} + 5C_{20})a\rho^2 + 4(B_{11} - C_{02} + C_{20})^2$, $g_2(\rho) = C_{20}(C_{20} + C_{02})a\rho^6 - (B_{11}C_{02} - C_{02}^2 + 2B_{11}C_{20} + 3C_{20}^2)\rho^4 + (B_{11} - C_{02} + C_{20})(B_{11} + 3C_{20})a\rho^2 - (B_{11} - C_{02} + C_{20})^2$ *and*

$$\Sigma(\rho) = \big(1 - a\rho^2\big)\big((C_{20} + C_{02})a\rho^2 + C_{02} - C_{20} - B_{11}\big). \tag{4.19}$$

Proof. Since $S_2(\theta) \equiv 0$ when (4.7) holds, from (2.6) given in Theorem 2.1 we obtain that

$$\begin{aligned} T_2(\rho) &= \int_0^{2\pi} \Big(\big(r_0(\theta,\rho)S_1(\theta) + r_0^2(\theta,\rho)S_2(\theta)\big)^2 - r_1(\theta,\rho)S_1(\theta) - 2r_0(\theta,\rho)r_1(\theta,\rho)S_2(\theta)\Big)\, d\theta \\ &= \int_0^{2\pi} \big(r_0^2(\theta,\rho)S_1^2(\theta) - r_1(\theta,\rho)S_1(\theta)\big)\, d\theta. \end{aligned} \tag{4.20}$$

It follows from (2.7) and (2.8) that

$$\begin{aligned} r_0(\theta,\rho) &= \rho\big(1 - a\rho^2\sin^2\theta\big)^{-1/2}, \\ r_1(\theta,\rho) &= \int_0^\theta \big\{r_0^2(\alpha,\rho)G_1(\alpha) + r_0^3(\alpha,\rho)G_2(\alpha) - r_0^4(\alpha,\rho)G_0(\alpha)S_1(\alpha)\big\} \\ &\quad \cdot \exp\left(\int_\alpha^\theta 3r_0^2(\beta,\rho)G_0(\beta)\, d\beta\right) d\alpha \\ &= \int_0^\theta \big\{r_0^2(\alpha,\rho)G_1(\alpha) + r_0^3(\alpha,\rho)G_2(\alpha) - r_0^4(\alpha,\rho)G_0(\alpha)S_1(\alpha)\big\}\left(\frac{1 - a\rho^2\sin^2\alpha}{1 - a\rho^2\sin^2\theta}\right)^{\frac{3}{2}} d\alpha \\ &= r_0^3(\theta,\rho)\int_0^\theta \left(\frac{G_1(\alpha)}{r_0(\alpha,\rho)} + G_2(\alpha) - r_0(\alpha,\rho)G_0(\alpha)S_1(\alpha)\right) d\alpha. \end{aligned}$$

Thus,

$$\begin{aligned} &\int_0^{2\pi} r_1(\theta,\rho)S_1(\theta)\, d\theta \\ &\quad = \int_0^{2\pi} \left\{ S_1(\theta)r_0^3(\theta,\rho)\int_0^\theta \left(\frac{G_1(\alpha)}{r_0(\alpha,\rho)} + G_2(\alpha) - r_0(\alpha,\rho)G_0(\alpha)S_1(\alpha)\right) d\alpha \right\} d\theta \end{aligned}$$

$$= \int_0^{2\pi} \left\{ S_1(\theta) r_0^3(\theta,\rho) \int_0^{\theta} \left(\frac{G_1(\alpha)}{r_0(\alpha,\rho)} - r_0(\alpha,\rho) G_0(\alpha) S_1(\alpha) \right) d\alpha \right\} d\theta$$

$$= \int_0^{2\pi} \left\{ S_1(\theta) r_0^3(\theta,\rho) \int_0^{\theta} \frac{(1-a\rho^2)C_{02}\sin^3\alpha + (B_{11}+C_{20}-C_{20}a\rho^2)\cos^2\alpha\sin\alpha}{\rho\sqrt{1-a\rho^2\sin^2\alpha}} d\alpha \right\} d\theta$$

$$= \frac{(1-a\rho^2)(C_{02}-C_{20}) - B_{11}}{2a\rho^2} \int_0^{2\pi} r_0^2(\theta,\rho) S_1(\theta)\cos\theta\, d\theta$$

$$- \begin{cases} \frac{\Sigma(\rho)}{2\rho}\int_0^{2\pi} \frac{S_1(\theta)\ln(\sqrt{1-\rho^2\sin^2\theta}+\rho\cos\theta)}{(1-\rho^2\sin^2\theta)^{3/2}} d\theta, & \rho\in(0,1) \text{ for } a=1, \\ \frac{\Sigma(\rho)}{-2\rho}\int_0^{2\pi} \frac{S_1(\theta)\arcsin(\frac{\rho\cos\theta}{\sqrt{1+\rho^2}})}{(1+\rho^2\sin^2\theta)^{3/2}} d\theta, & \rho\in(0,K_1) \text{ for } a=-1, \end{cases}$$

$$= \frac{-\pi}{2a\rho^4} f_1(\rho) + \frac{\pi}{a\rho^4\sqrt{1-a\rho^2}} f_2(\rho)$$

$$- \begin{cases} \frac{\Sigma(\rho)}{\rho}\int_0^{\pi} \frac{S_1(\theta)\ln(\sqrt{1-\rho^2\sin^2\theta}+\rho\cos\theta)}{(1-\rho^2\sin^2\theta)^{3/2}} d\theta, & \rho\in(0,1) \text{ for } a=1, \\ \frac{\Sigma(\rho)}{-\rho}\int_0^{\pi} \frac{S_1(\theta)\arcsin(\frac{\rho\cos\theta}{\sqrt{1+\rho^2}})}{(1+\rho^2\sin^2\theta)^{3/2}} d\theta, & \rho\in(0,K_1) \text{ for } a=-1, \end{cases} \tag{4.21}$$

where $\Sigma(\rho)$ is given by (4.19) and $f_1(\rho) = (C_{02}-C_{20})(B_{11}-C_{02}+3C_{20})\rho^4 + (B_{11}-C_{02}+C_{20})(B_{11}-3C_{02}+5C_{20})a\rho^2 - 2(B_{11}-C_{02}+C_{20})^2$, $f_2(\rho) = C_{20}(C_{20}-C_{02})a\rho^6 + (B_{11}C_{02} - C_{02}^2 - 2B_{11}C_{20} + 4C_{02}C_{20} - 3C_{20}^2)\rho^4 + (B_{11}-C_{02}+C_{20})(B_{11}-2C_{02}+3C_{20})a\rho^2 - (B_{11}-C_{02}+C_{20})^2$.

On the other hand,

$$\int_0^{2\pi} r_0^2(\theta,\rho) S_1^2(\theta)\, d\theta$$

$$= \rho^2 \int_0^{2\pi} \frac{C_{20}^2\cos^6\theta + (C_{02}-B_{11})^2\cos^2\theta\sin^4\theta + 2C_{20}(C_{02}-B_{11})\cos^4\theta\sin^2\theta}{1-a\rho^2\sin^2\theta} d\theta$$

$$= \frac{-\pi}{4a\rho^4}\big((3A+4B+8D)\rho^4 + a(4A+8B)\rho^2 + 8A\big)$$

$$+ \frac{2\pi}{a\rho^4\sqrt{1-a\rho^2}}\big(aC_{20}^2\rho^6 + D\rho^4 + aB\rho^2 + A\big), \tag{4.22}$$

where $A = -(B_{11}-C_{02}+C_{20})^2$, $B = (B_{11}-C_{02}+C_{20})(B_{11}-C_{02}+3C_{20})$ and $D = -C_{20}(2B_{11} - 2C_{02}+3C_{20})$. It follows from (4.20), (4.21) and (4.22) that $T_2(\rho)$ has the expression in (4.18). □

By Theorem 4.2 we obtain the expression of $T_2(\rho)$ for system (4.1) in the following corollary.

Corollary 4.2. *Assume that* (4.16) *holds but* (4.15) *does not hold for system* (4.1)*. Then the first PBF* $T_1(\rho)\equiv 0$ *and the second one*

$$T_2(\rho) = \frac{-\pi}{4a_{21}^3\rho^4} g_1(\rho) + \frac{\pi}{a_{21}^3\rho^4\sqrt{1-a_{21}\rho^2}} g_2(\rho)$$
$$+ \begin{cases} \frac{\sqrt{a_{21}}\Sigma(\rho)}{\rho a_{21}^2} \int_0^\pi \frac{S_1(\theta)\ln(\sqrt{1-a_{21}\rho^2\sin^2\theta}+\sqrt{a_{21}}\rho\cos\theta)}{(1-a_{21}\rho^2\sin^2\theta)^{3/2}}\, d\theta, & \rho \in (0, \frac{1}{\sqrt{a_{21}}})\ \textit{for}\ a_{21} > 0, \\ \frac{\sqrt{-a_{21}}\Sigma(\rho)}{-\rho a_{21}^2} \int_0^\pi \frac{S_1(\theta)\arcsin(\frac{\sqrt{-a_{21}}\rho\cos\theta}{\sqrt{1-a_{21}\rho^2}})}{(1-a_{21}\rho^2\sin^2\theta)^{3/2}}\, d\theta, & \rho \in (0, K_2)\ \textit{for}\ a_{21} < 0, \end{cases} \tag{4.23}$$

where $S_1(\theta) = c_{20}\cos^3\theta + (c_{02} - b_{11})\cos\theta\sin^2\theta$, $\Sigma(\rho) = (1 - a_{21}\rho^2)((c_{20} + c_{02})a_{21}\rho^2 + c_{02} - c_{20} - b_{11})$, $g_1(\rho) = (-b_{11}^2 + 4b_{11}c_{02} - 3c_{02}^2 + 4b_{11}c_{20} + 2c_{02}c_{20} + 9c_{20}^2)a_{21}^2\rho^4 - 2(b_{11} - c_{02} + c_{20})(b_{11} + c_{02} + 5c_{20})a_{21}\rho^2 + 4(b_{11} - c_{02} + c_{20})^2$ *and* $g_2(\rho) = c_{20}(c_{20} + c_{02})a_{21}^3\rho^6 - (b_{11}c_{02} - c_{02}^2 + 2b_{11}c_{20} + 3c_{20}^2)a_{21}^2\rho^4 + (b_{11} - c_{02} + c_{20})(b_{11} + 3c_{20})a_{21}\rho^2 - (b_{11} - c_{02} + c_{20})^2$.

Proof. We only give the proof for the case $a_{21} > 0$. The proof for the case $a_{21} < 0$ can be obtained similarly. We need to apply Theorem 4.2 to system (4.2) associated with (4.4). With the restriction (4.4), the second PBF, given in Theorem 4.2, can be expressed as

$$\tilde{T}_2(\rho) = \frac{-\pi\hat{g}_1(\rho)}{4a_{21}\rho^4} + \frac{\pi\hat{g}_2(\rho)}{a_{21}\rho^4\sqrt{1-\rho^2}} + \frac{\hat{\Sigma}(\rho)}{a_{21}\rho}\int_0^\pi \frac{\hat{S}_1(\theta)\ln(\sqrt{1-\rho^2\sin^2\theta}+\rho\cos\theta)}{(1-\rho^2\sin^2\theta)^{3/2}}\, d\theta,$$

where $\hat{S}_1(\theta) = c_{20}\cos^3\theta + (c_{02} - b_{11})\cos\theta\sin^2\theta$, $\hat{\Sigma}(\rho) = (1-\rho^2)((c_{20} + c_{02})\rho^2 + c_{02} - c_{20} - b_{11})$, $\hat{g}_1(\rho) = (-b_{11}^2 + 4b_{11}c_{02} - 3c_{02}^2 + 4b_{11}c_{20} + 2c_{02}c_{20} + 9c_{20}^2)\rho^4 - 2(b_{11} - c_{02} + c_{20})(b_{11} + c_{02} + 5c_{20})\rho^2 + 4(b_{11} - c_{02} + c_{20})^2$ and $\hat{g}_2(\rho) = c_{20}(c_{20} + c_{02})\rho^6 - (b_{11}c_{02} - c_{02}^2 + 2b_{11}c_{20} + 3c_{20}^2)\rho^4 + (b_{11} - c_{02} + c_{20})(b_{11} + 3c_{20})\rho^2 - (b_{11} - c_{02} + c_{20})^2$.

Note that the rescaling (4.3), which changes system (4.1) into system (4.2) with $a = 1$ and (4.4), gives the correspondence $(\rho, 0) \mapsto (\sqrt{a_{21}}\rho, 0)$ from the (x, y)-space to the (u, v)-space. Thus, for system (4.1) with $a_{21} > 0$ we obtain the second PBF $T_2(\rho) = \tilde{T}_2(\sqrt{a_{21}}\rho)$, which exactly is the expression in (4.23) for $a_{21} > 0$. $\square$

When (4.16) holds but (4.15) does not hold, the explicit expression of $T_2(\rho)$ for system (4.1) is not given in Corollary 4.2 because we are not able to compute the integral on the right-hand side of (4.23). When (4.16) holds and $b_{11} + 2c_{20} = c_{02} + c_{20} = 0$, system (4.1) is written as

$$\begin{cases} \dot{x} = -y + a_{21}x^2y + \varepsilon(-2c_{20}xy + b_{21}x^2y), \\ \dot{y} = x + a_{21}xy^2 + \varepsilon(c_{20}x^2 - c_{20}y^2 + b_{21}xy^2). \end{cases} \tag{4.24}$$

By Corollary 4.2 we obtain that $T_2(\rho) \equiv 0$ for system (4.24) because $g_1(\rho) \equiv g_2(\rho) \equiv \Sigma(\rho) \equiv 0$. On the other hand, from [29] we see that center O of (4.24) is isochronous. Thus, $T_i(\rho) \equiv 0$ for $i = 1, \ldots$. This does not contradict to Corollary 4.2.

Acknowledgments

The authors are grateful to referees for their helpful comments and suggestions. The first author thanks the Center for Applied Mathematics and Theoretical Physics, University of Maribor, for hospitality when he visited there.

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